ON A METHOD OF HOLOPAINEN AND RICKMAN

BY

DAVID DRASIN

Department of Mathematics, Purdue University
West Lafayette, IN 47907, USA
e-mail: drasin@math.purdue.edu

ABSTRACT

There exists a quasiregular map on $\mathbb{R}^n$ $(n \geq 3)$ of finite order for which every $a \in \mathbb{R}^n$ is an asymptotic value.

1. Introduction

Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be quasiregular (qr). Thus $f \in W^1_{n,loc}(\mathbb{R}^n)$ and for some $K \geq 1$, $|f'(x)|^n \leq KJ_f(x)$ a.e.; here f' is the formal derivative of $f, |f'(x)|$ is the operator norm, and J_f the Jacobian determinant. Standard references are [6] and [7]. The order of f is

$$\lambda = \limsup_{r \longrightarrow \infty} (n-1)\frac{\log\log M(r)}{\log r},$$

with $M(r) = \max_{|x|=r} |f(x)|$.

A number $a \in \mathbb{R}^n$ is an **asymptotic value** of f if $f(x) \longrightarrow a$ as $|x| \longrightarrow \infty$ on some path $\gamma \subset \mathbb{R}^n$. In [5], I. Holopainen and S. Rickman constructed a qr map f on $\mathbb{R}^n$, $n \geq 3$, with $\lambda = n-1$ and countably many asymptotic values. Here we modify (and, in some ways, simplify) that construction to prove

THEOREM 1.1: *There exists an entire qr map f on $\mathbb{R}^n$ $(n \geq 3)$ of order $n-1$ with every $a \in \mathbb{R}^n$ asymptotic.*

When $n = 2$, Ahlfors's theorem asserts that the number of distinct asymptotic values of an analytic entire function of order λ is at most 2λ, and it follows that

Received December 12, 1994 and in revised form August 26, 1996

the number is bounded for qr maps of finite order on $\mathbb{R}^2$ as well; see the discussion in [5].

However, if $\lambda = \infty$, W. Gross [4] has constructed an entire function with every value a asymptotic, and if f is allowed to have poles, A. Eremenko [3] has produced a meromorphic function with every a asymptotic and $T(r,f)(\log r)^{-2}$ tending to infinity as slowly as desired. ($T(r,f)$ is the Nevanlinna characteristic.)

The distinction between $\mathbb{R}^2$ and $\mathbb{R}^n$ ($n \geq 3$) is that the asymptotic curves do not partition the domain in higher dimensions.

As in [5], this construction is a generalization of the entire function $\sin z/z$, with $\rho = 1$ and for which there are two asymptotic tracts (the positive/negative axes). These correspond to the common asymptotic value $a = 0$. The image of the asymptotic curve passes through 0 infinitely often, through periods of progressively smaller amplitude.

Note that if f has an asymptotic value, then its order (lower order) must be $> c(n,K) > 0$ [8].

We use standard notation: $x \in \mathbb{R}^n$ is written $x = (x_1, \ldots, x_n) = (x', x_n)$, and we identify $\mathbb{R}^{n-1}$ with $\{x \in \mathbb{R}^n; x_n = 0\}$. Also, $B(a,R) = \{|x-a| < R\}$, $B(R) = B(0,R)$, $B = B(1)$, $S(a,R) = \partial B(a,R)$, $S(R) = \partial B(R)$, $S = \partial B$. For $a \in \mathbb{R}^n$, let

$$Q(a,h) = \{x \in \mathbb{R}^n; |x_j - a_j| < h, 1 \leq j \leq n\}.$$

If $x = (x', x_n) \in \mathbb{R}^n$, $\overline{x} = (x', -x_n)$, and for $E \subset \mathbb{R}^n$, $\overline{E} = \{x; \overline{x} \in E\}$. When these notions are applied to sets in $\mathbb{R}^{n-1}$, the corresponding sets are $B'(a,R), B'(R), \ldots$. Finally, $A'(r,s) = \{x' \in \mathbb{R}^{n-1}; r < |x'| < s\}$, $\|x'\|^* = \max_{1 \leq j \leq n-1} |x_j|$, and if $E \subset \mathbb{R}^{n-1}$, $F \subset \mathbb{R}^{n-1}$, $d'(E,F)$ is their $(n-1)$-distance.

ACKNOWLEDGEMENT: I thank Juha Heinonen for several useful discussions and detecting an error in my first version, and the authors of [5] for an advance copy. I am also grateful to the referee for an unusually careful scrutiny that I trust is reflected in what appears now.

2. A qr sine function

The procedure is a bit different from that of [5], in that we are more imitative of the classical sine function, rather than the (Zorich) exponential function, and we rely systematically on compositions with simple qc homeomorphisms. If $S(x)$ is qr and $H: \mathbb{R}^n \longrightarrow \mathbb{R}^+$ is smooth, the product HS in general is not qr. For it to be qr with a carefully-chosen H, we shall see in Lemma 5.4 that S is forced not only to be qr, but be *locally linearly nondegenerate* in the sense that there exists $c > 0$ so that for a.e. x

$$\frac{|S(x+p)-S(x)|}{|p|} > c|S(x)| > 0, \quad |p| < p_0(x). \tag{2.1}$$

Many qr maps satisfy conditions like (2.1); a familiar example is the winding map [7, p. 13] near $x = 0$. Nothing like (2.1) holds in general for analytic functions near points of ramification; in particular (2.1) fails for $\sin z$ itself.

As suggested in [7, p. 15], divide $\mathbb{R}^n$ into congruent cylinders C by means of the hyperplanes $H_{j,k}$: $x_j = \frac{1}{2} + k$, $1 \le j \le n-1$, $k \in \mathbb{Z}$. Let C_0 be that which contains 0. Let $C_0^* = \{x \in C_0; x_n > \frac{1}{2} - \|x'\|^*\}$, let $c_0 = \sinh^{-1}(1)$, and we will define a K-qc, L-bilipschitz map

$$g\colon C_0^* \longrightarrow V^+ = \{x \in \mathbb{R}^n; |x'| < \pi/2, x_n > c_0\}$$

with $g = g_2 \circ g_1$. Here $g_1(x', x_n) = (x', x_n + \|x'\|^* - \frac{1}{2})$ maps C_0^* onto $C_0^+ = \{x \in C_0; x_n > 0\}$, and $g_2(x', x_n) = (k(x'), x_n + c_0)$, where k is a K-qc locally L-bilipschitz map of $Q'(0, \frac{1}{2})$ onto $B'(\pi/2)$, maps C_0^+ to V^+.

The map $h\colon V^+ \longrightarrow \mathbb{R}^n$ with

$$h(y', y_n) = (\zeta', \zeta_n) = \left(\frac{y'}{|y'|}\sin|y'|\cosh y_n,\ \cos|y'|\sinh y_n\right) \tag{2.2}$$

sends $V^+ \cap \{y_n = c\}$, $c > c_0$, onto the upper half of the ellipsoid $E(\cosh c,\ \sinh c)$ of height $\sinh c$, base $B'(\cosh c)$, and this mapping is qc on V^+. Indeed let $X = \{z = x + iy \in \mathbb{C}; y > c_0\}$. Then for each fixed $x' \in S'$, the function $s\colon X \longrightarrow \mathbb{R}^n$ given by

$$\begin{aligned} s(x,y) &= h(xx', y) \\ &= ((\operatorname{sgn} x \sin|x|\cosh y)x',\ \cos|x|\sinh y) \\ &= (\sin x \cosh y)x' + (\cos x \sinh y)x_n \end{aligned}$$

becomes the usual sine function when $n = 2$, $x' = 1$, $x_2 = i$. That h is qr now follows as in [6, p. 65]. The composition

$$S = h \circ g \tag{2.3}$$

is qr and maps C_0^* onto $\mathbb{R}^n_+ \backslash E(\cosh c_0, 1)$.

The function S is next extended to all of C_0. First, for $x \in \overline{C_0^*}$ set $S(x) = S(\overline{x})$. The set

$$T = C_0 \backslash (C_0^* \cup \overline{C_0^*})$$

is a lipschitz polyhedron of height 1, and S maps ∂T onto $E = E(\cosh c_0, \sinh c_0\}$ $= E(\cosh c_0, 1)$ in a bilipschitz manner. S then may be extended into T as a K-qc, L-bilipschitz homeomorphism onto E with $S(\overline{x}) = \overline{S(x)}$, $S(0) = 0$.

This defines S on all of C_0, and since $S\colon \partial C_0 \longrightarrow \mathbb{R}^{n-1}$, S may be repeatedly reflected across the faces of the various cylinders C to be defined on all $\mathbb{R}^n$. The branch set consists of the $(n-2)$-cells which are common to at least three faces of the cylinders C, together with $\{\bigcup_C \partial C\} \cap \mathbb{R}^{n-1}$. Note that if $x \in \mathbb{R}^n$, $x_n = c \geq c_0$, then

$$|\sinh c| \leq |S(x)| \leq \cosh c.$$

We call S a qr sine function: S is periodic (period 1) in each of the first $n-1$ variables, $|S| \longrightarrow \infty$ uniformly as $|x_n| \longrightarrow \infty$, and $S(0) = 0$.

LEMMA 2.4: *S is qr of order $n-1$, and there exists $c > 0$ such that* (2.1) *holds.*

Proof: That $\lambda = n - 1$ follows from the above estimate of $|S(x)|$ and the definition of order, and we have already seen that S is qr. We next consider (2.1).

Let $y \in V^+$ and let P be (a/the) two-dimensional plane through y and the $x' = 0$ axis. The complex function $\sin z$ is analytic with $(\sin z)' = \cos z$, and $|\sin z| > 1$, $|\cos z| > 1$ when $|\mathrm{Im}\, z| > c_0 > 0$. Hence if also $p \in P$, $|p| < p_0(y)$, we find from (2.2) and the fact that $y_n > 1$ that

$$|h(y+p) - h(y)| \geq a_1 |p| |h(y)| \quad (p \in P,\ |p| < p_1(y)),$$

and since h is K_1-qc at y, we deduce that

$$|h(y+p) - h(y)| > a|p||h(y)| \quad (|p| < p_0(y)), \tag{2.5}$$

where $a = a(a_1, K_1, n)$.

Since g is bilipschitz, (2.1) follows from (2.3) and (2.5) for $x \in C_0^* \cup \overline{C_0^*}$:

$$\begin{aligned} |S(x+p) - S(x)| &= |h(g(x+p)) - h(g(x))| \\ &\geq \tfrac{1}{2} a |g(x+p) - g(x)| S(x) \\ &\geq c|p| S(x) \qquad (|p| < p_0). \end{aligned}$$

In addition, S is bilipschitz in T with $|S(x)| \leq \cosh c_0$, so (2.1) again holds for $x \in T$ for some $c > 0$. Finally, we obtain (2.1) for all x by noting that S has been extended to all of $\mathbb{R}^n$ by reflection on the faces of the cylinders C.

Remarks 2.6: If we were to use (2.2) on all of $V = \{x \in \mathbb{R}^n; |x'| < \pi/2\}$ rather than V^+, estimate (2.1) would fail near the points of $\partial V \cap \mathbb{R}^{n-1}$. Note also that if $0 < \lambda \leq 1$, each set

$$W_\lambda = \{x \in \mathbb{R}^n,\ |S(x)| < \lambda\} \tag{2.7}$$

is a disjoint union of n-cells, symmetric with respect to $\mathbb{R}^{n-1}$, one compactly contained in each cylinder C, and to each of which may be associated a point $(j, 0) \in \mathbb{Z}^{n-1} \times \{0\}$ as center.

3. A forest of trees

First, let L be a tree which for each $k \geq 1$ contains 2^{kn} edges of generation k, each of which is attached to one edge of order $k-1$ and to 2^n of order $k+1$; generation 0 is the common initial point of the 2^n first-generation edges. Let S be the cell

$$S = \{x \in \mathbb{R}^n; 0 \leq x_i \leq 1, 1 \leq i \leq n\}. \tag{3.1}$$

Then to each $a \in S$ is associated a path $L(a)$ in L whose edge of generation k corresponds to the approximation to a of the first k binary digits in each coordinate:

$$a_k = (.a_1^1 a_2^1 \dots a_k^1,\ \dots\ ,\ .a_1^n a_2^n \dots a_k^n) \tag{3.2}$$

with each $a_m^i \in \{0, 1\}$. This correspondence is coherent in the sense that if an edge of generation k joins one of generation $k+1$, then $a_k - a_{k+1} \in Q(0, 2^{-k})$.

Consider the integral points $j = (j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$. The rank of j is the integer $\|j\| = N = \max |j_k|$. To each $j \in \mathbb{Z}^n$ will correspond the closed cell

$$S_j = \{x \in \mathbb{R}^n;\ 0 \leq x_i - j_i \leq 1,\ 1 \leq i \leq n\} \tag{3.3}$$

in $\mathbb{R}^n$ and a tree L_j will be associated to j. Each L_j is combinatorially equivalent to L, and each number $b \in S_j$, where $b = a + j,\ a \in S$, corresponds to the path

$L_j(b)$ which is combinatorially congruent to the $L(a)$ described above. Note that $\bigcup \overline{S_j} = \mathbb{R}^n$. We say L_j is of rank N if $\|j\| = N$.

Each L_j is realized geometrically in a unique region $D_j \subset \mathbb{R}^{n-1}$, such that each D_j and D_k are congruent by a rigid motion of $\mathbb{R}^{n-1}$. Each D_j is a paraboloid region, with

$$D_0 = \{x'; x_2^2 + \cdots + x_{n-1}^2 < x_1\},$$

and D_j is said to be of rank N if N is the rank of j. We construct a rapidly-increasing sequence $\{r_M\}$, with $r_0 = 0$, such that the vertex of each D_j of rank N lies on $S'(r_N)$ and $D_j \subset \{|x'| \geq r_N\}$. The $\{r_M\}$ are chosen so that if D_j and D_k meet $S(r)$ and d is Euclidean distance, then

$$d(D_j \cap S'(r), D_k \cap S'(r)) \geq 2d_M \geq 1 \quad (r \geq r_M).$$

By choosing the $\{r_M\}$ appropriately, the $d_M \uparrow \infty$ as fast as desired. In §5 we impose conditions which force the d_M to increase rapidly.

Each L_j of rank N is placed in $D_j \cap \{|x'| \geq r_{N+1}\}$. Those edges of L_j of generation k, $L_j^{(k)}$ are realized as straight lines in $\mathbb{R}^{n-1}$ contained in $A'(r_{N+k}, r_{N+k+1})$ such that if ℓ and ℓ' are of the kth generation but have different ancestors in $L_j^{(k-1)}$, then

$$d(\ell, \ell') \geq 2d_{N+k} \tag{3.4}$$

and

$$d(\ell, \partial D_j) \geq 2d_{N+k}. \tag{3.5}$$

Note that only a fixed number of D_j meet any $A'(r_M, r_{M+1})$, so (3.4) and (3.5) can be ensured if r_{M+1}/r_M is large.

4. Near-Möbius mappings

Our function f depends on two classes of quasiconformal mappings, which we call φ_0 and φ_k ($k \geq 1$). Our first result will yield φ_0, and the others will follow from Lemma 4.2.

LEMMA 4.1: *Given $K > 1, N \geq 0, \delta > 0$, there exists $R = R(K, N, \delta, n)$ such that to each $b \in Q(0, N+1)$ corresponds a K-qc map φ of $\mathbb{R}^n$ with*

$$\begin{aligned} \varphi(w) &= w && (|w| > R), \\ &= w + b && (w \in Q(0, \delta)). \end{aligned}$$

LEMMA 4.2: *Given $K > 1, R > 0$, there exists $\delta = \delta(K, R, n) > 0$ such that to each $a \in Q(0, \delta)$ corresponds a K-qc map ψ of $\mathbb{R}^n$ with*

$$\begin{aligned} \psi(w) &= w && (|w| > R), \\ &= w + a && (w \in Q(0, \delta)). \end{aligned}$$

(The proofs are routine; for a proof in the two-dimensional case, see [1] or [2].)

Remark 4.3: For a given $K > 1$, choose $K_0, K_1, \ldots > 1$ with

$$\prod_0^\infty K_k < K. \tag{4.4}$$

Then Lemma 4.2 is used with elementary normal family considerations.

(A) For $k \geq 1$ let the $\{R_k\}, \{\delta_k\}$ be bounded, let $\{b_k\}_0^\infty$ be a bounded sequence. Let ψ_k be chosen as in Lemma 4.2 with data $K = K_k, R = R_k, \delta = \delta_k, a = b_k - b_{k-1}$, where we suppose that

$$b_k - b_{k-1} \in Q(0, \delta_k). \tag{4.5}$$

Let φ_k satisfy

$$\varphi_k(w) = \psi_k(w - b_{k-1}) + b_{k-1}, \tag{4.6}$$

so that

$$\begin{aligned} \varphi_k(w) &= w && (|w - b_{k-1}| > R_k), \\ &= w + b_k - b_{k+1} && (w \in Q(b_{k-1}, \delta_k)). \end{aligned} \tag{4.7}$$

If we write

$$\Phi_k = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1 \qquad (k \geq 1), \tag{4.8}$$

then the Φ_k form a normal family of K-qc homeomorphisms.

(B) If in addition $\sum R_k < \infty$ (so that $\sum \delta_k < \infty$), then the $\{\Phi_k\}$ tend normally to a qc homeomorphism.

Lemmas 4.1 and 4.2 give, for each $N \in \mathbb{Z}^+$, sequences

$$R_0(N) > R_1 > \cdots > R_k > \cdots, \quad R_k \to 0 \quad (k \to \infty),$$

and δ_k ($k \geq 0$) in the following way. First take $\delta_0 = 1$ and for each $N \geq 0$ choose $R_0 = R_0(N)$ according to Lemma 4.1 with $K = K_0$, K_0 from (4.4). Then for $k \geq 1$, take inductively R_k so that $0 < R_k < \min(R_{k-1}, 2^{-k})$ and

$$B(6R_k) \subset Q(0, \delta_{k-1}), \tag{4.9}$$

and then $K = K_k$ and δ_k from (4.4) and Lemma 4.2 respectively. The mappings φ_k will be constructed in §5 with specific $b = b_k$, using (4.6) when $k \geq 1$. Note that $R_0(N) \to \infty (N \to \infty)$ but that $\sum_1^\infty R_k < 1$; thus alternative (B) will apply to any of our families $\{\Phi_k\}$.

Finally, we introduce a subsequence of integers $\{k_p\}_{p \geq 0}$ with

$$2^{-k_p} \leq \delta_{p+1}. \tag{4.10}$$

5. Proof of Theorem 1.1

With $\varepsilon_0 > 0$ to be chosen, let $H\colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^+$ be smooth such that

$$H(x') = 1 \quad (Q(x',1) \cap (\cup D_j) = \emptyset), \tag{5.1}$$

$$H(x') \leq 1 + |x'| \qquad (x' \in \mathbb{R}^{n-1}), \tag{5.2}$$

and

$$|\nabla \log H(x')| < \varepsilon_0 \qquad (x' \in \mathbb{R}^{n-1}). \tag{5.3}$$

LEMMA 5.4: *If ε_0 is sufficiently small in* (5.3), *$S(x)$ is from* §2, *and $x = (x', x_n)$, then*

$$f_0(x) = H(x')S(x)$$

is qr on $\mathbb{R}^n$, of order n-1.

Proof: Let x and p be in $\mathbb{R}^n$. Then

$$\begin{aligned} \Delta f_0 &= f_0(x+p) - f_0(x) \\ &= S(x+p)(H(x+p) - H(x)) + H(x)(S(x+p) - S(x)) \\ &= S\Delta H + H\Delta S. \end{aligned}$$

By Lemma 2.4, $H|\Delta S| > cH|S||p|$ ($|p| < p_0$), so that if $\varepsilon_0 < \frac{1}{4}c$, (5.3) implies that

$$\begin{aligned} ||\Delta f_0| - H|\Delta S|| = |S|\,|\Delta H| &< 2\varepsilon_0\,|p|H|S| \\ &< \tfrac{1}{2}H|\Delta S| \qquad (|p| < p_0). \end{aligned}$$

Since S is K-qc, it follows that if x is fixed and $|h| < h_0(x)$, then

$$\begin{aligned}\sup_{|p|=h} |\Delta f_0| &\le \tfrac{3}{2} H(x) \sup_{|p|=h} |\Delta S| \\ &\le \tfrac{3}{2} H(x) K \inf_{|p|=h} |\Delta S| \\ &\le 3K \inf_{|p|=h} |\Delta f_0|,\end{aligned}$$

so by [7, p. 42] f_0 is qr. That f_0 has order $n-1$ follows from (5.1), (5.2) and Lemma 2.4.

Now let $j \in \mathbb{Z}^n$ be fixed with $\|j\| = N$, and $D_j \subset \mathbb{R}^{n-1}$ and $\{d_M\}$ be as at the end of §3. In D_j we construct sets $D_j(p)$ $(p \ge 0)$ with $D_j(p) \subset \{|x'| > r_{N+k_p+1}\}$ and

$$\cdots D_j(p) \subset D_j(p-1) \subset \cdots \subset D_j(0) \subset D_j,$$

where the k_p are from (4.10). Thus for $m \ge N + k_p + 1$, let

$$D_j(p) \cap A'(r_m, r_{m+1}) = \{x \in D_j \cap A'(r_m, r_{m+1}),\ d'(x, L_j) < d_m/(p+2)\}; \tag{5.5}$$

by (3.4) and (3.5) we have 2^{nk_p} components $D_j(p)$. Below we will usually think of j as fixed and usually write $D(p)$ for $D_j(p)$. The value of the $\{D(p)\}$ is that if $b = a + j$ and $b' = a' + j$ are associated to two edges of $L_j \cap D(p)$ for a single component $D(p)$, then by (3.2) and (4.10),

$$b - b' = a - a' \in Q(0, 2^{-k_p}) \subset Q(0, \delta_{p+1}). \tag{5.6}$$

We may slightly perturb the $D(p)$ so each $\partial D(p)$ is composed of portions of

$$\mathbb{R}^{n-1} \cap \{\bigcup \partial C\},$$

where the C's are the cylinders introduced in §2.

Now let us construct f. In order to achieve this, we will be forced to have the ratios d_{M+1}/d_M (and hence r_{M+1}/r_M in §3) increase rapidly.

If $x' \in \mathbb{R}^{n-1}$, let $C = C(x')$ be a cylinder C which contains x'. We demand, in addition to (5.1)–(5.3) that for each j

$$3R_0(N) > H(x') > 2R_0(N), \tag{5.7}$$

if $\|j\| = N$ and $C(x') \cap \partial D_j(0) \ne \emptyset$. We then set

$$f(x) = H(x')S(x) \quad (C(x') \cap \{\cup_j D_j(0)\} = \emptyset). \tag{5.8}$$

While (5.7) asserts that H is large near each $\partial D_j(0)$, we now force $H(x')$ to become small as $x' \longrightarrow \infty$ in $D_j(0)$ near L_j. Thus, given the $\{R_k\}(k \geq 1)$ from §4, we require that

$$(5.9) \qquad 2R_p < H(x') < R_{p-1} \quad (C(x') \cap \partial D_j(p) \neq \emptyset),$$

$$(5.10) \qquad H(x') < R_{p-1} \quad (x' \in D_j(p)).$$

Note that (5.9) and (5.10) do not depend on $N = \|j\|$, and that all of (5.7), (5.9) and (5.10) are possible if the ratios $d_{M+1}/d_M \longrightarrow \infty$ sufficiently rapidly in §3.

Now we augment (5.8) and define f in each $\bigcup D(0)\backslash \bigcup(D(1))$; we have observed that there are 2^{k_0} such components inside each D_j. Recall the graph $L_j \subset D_j$. Then to each $D_j(0)$ we choose $b_0 = j + a_0$ (with $a_0 \in S$) which corresponds to some edge of $L_j \cap D(0)$ as in (3.2). While b_0 is far from unique, we have (5.6) with $p = 0$. Thus, to each component $D(0)$, choose φ_0 from Lemma 4.1 with $K = K_0$, $\delta_0 = 1$ and $b = b_0 = j + a_0 \in Q(0, N+1)$, and set

$$(5.11) \qquad f(x) = \varphi_0(H(x')S(x)) \quad (x \in D(0)\backslash \bigcup D(1)).$$

In general, given $p \geq 1$ and $D(p) \subset D_j$, then $D(p)$ is nested in a family $D(k)$, $0 \leq k < p-1$, each contained in D_j. Suppose f has been defined in each $D(p-1)$ by

$$(5.12) \qquad f(x) = \varphi_{p-1} \circ \varphi_{p-2} \circ \cdots \circ \varphi_0(H(x')S(x)) \quad (x \in D(p-1)\backslash \bigcup D(p)),$$

where φ_0 has just been described, and the φ_k $(1 \leq k \leq p-1)$ have been selected in accord with Lemma 4.2 and (4.6) with $R = R_k$, $\delta = \delta_k$, $K = K_k$ and $a_k \in Q(0, \delta_k)$ such that $b_k = j + a_k$ corresponds to a corresponding edge of $L_j \cap D(k)$. Then to each $D(p)$, we associate a_p such that $j + a_p$ corresponds to an edge of $L_j \cap D(p)$ as in §3. With $a = a_p$, $b_p = j + a_p$, choose a K_p-qc homeomorphism from (4.6) (now with $k = p$), and set

$$(5.13) \qquad f(x) = \varphi_p \circ \varphi_{p-1} \circ \ldots \circ \varphi_0(H(x')S(x)) \quad (x \in D(p)\backslash \cup D(p+1)).$$

This with (5.8), (5.11) and (5.12) defines f on $\mathbb{R}^n$.

What is crucial is to see that f is continuous; once that is settled, (4.4), (5.8) and (5.13) show f is K_1-qr for some $K_1 = K_1(K, n)$. But S, H and all φ's are continuous, so we need only check continuity near each $\partial D(p)$, when the factor

φ_p is introduced as from (5.12) to (5.13). Thus, let $C(x') \cap \partial D(p) \neq \emptyset$. Then (5.9) and (2.7) imply that $H(x')|S(x)| > R_p$. Our choices (4.9) and Lemmas 4.1 and 4.2 imply that $R_p < \delta_{p-1} < \cdots < \delta_1 < \delta_0$ and hence Lemmas 4.1 and 4.2 (see (4.6) and (4.7)) show that the orbit of $B(0, R_p)$ under $\varphi_{p-1} \circ \cdots \circ \varphi_0$ is

$$B(0,R_p) \xrightarrow{\varphi_0} B(b_0,R_p) \xrightarrow{\varphi_1} B(b_1,R_p) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{p-1}} B(b_{p-1},R_p), \tag{5.14}$$

and on $B(0, R_p)$ this composition is the translation $w \longrightarrow w + b_{p-1}$. Hence (4.7) shows that definitions (5.12) and (5.13) coincide and so f is continuous.

We now prove the Theorem. Let $b = a + j \in S_j$, with $\|j\| = N$. The curve $L(b)$ on which $f(x) \longrightarrow b$ has been described in §3: in each $A'(r_{N+m}, r_{N+m+1})$, $k_p \leq m < k_{p+1}$, it is a segment $\ell \subset L_j \cap D(p)$. So by (5.10) and (2.7), $H(x')|S(x)| < R_{p-1}$. Thus as $x \to \infty$, $x \in L(b)$, (5.10) shows that $H(x')S(x) \to 0$. And by assumption $\sum R_p < \infty$, so by (B) of Remark 4.3 the Φ_k in (4.8) tend to a qc-homeomorphism Φ. However, since $b_{p-1} \to b$ on $L(b)$, (4.7), (4.8) and (5.14) show that

$$\Phi(b_0) = b.$$

Hence as $x \to \infty, x \in L(b)$, we deduce from (5.11) and (5.14) that

$$\begin{aligned} f(x) &\to \Phi \circ \varphi_0(H(x')S(x)) \to \Phi \circ \varphi_0(0) \\ &= \Phi(b_0) = b. \end{aligned}$$

This completes the proof.

References

[1] D. Drasin, *The inverse problem of the Nevanlinna theory*, Acta Mathematica **138** (1977), 83–151.

[2] D. Drasin and W.K. Hayman, *Value distribution of functions meromorphic in an angle*, Proceedings of the London Mathematical Society (3) **48** (1984), 319–340.

[3] A. Eremenko, *The set of asymptotic values of a meromorphic function of finite order* (in Russian), Matematicheskie Zametki; English translation: Mathematical Notes **24** (1979), 914–916.

[4] W. Gross, *Eine ganz Funktion, für die jede komplexe Zahl Konvergenzwert ist*, Mathematische Annalen **79** (1918), 201–208.

[5] I. Holopainen and S. Rickman, *The problem of the Denjoy theorem for quasiregular maps in dimension* $n \geq 3$, Proceedings of the American Mathematical Society **124** (1996), 1783–1788.

[6] Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion* (in Russian), Nauka, Moscow, 1982; English translation: Translations of Mathematical Monographs 73, American Mathematical Society, Providence, RI, 1989.

[7] S. Rickman, *Quasiregular Mappings*, Ergebnisse der Mathematik (3) Band 26, Springer-Verlag, Berlin, 1993.

[8] S. Rickman and M. Vuorinen, *On the order of quasiregular mappings*, Annales Academicae Scientarum Fennicae, Series AI Mathematics **7** (1982), 221–231.